DETERMINATION OF DYNAMIC CHARACTERISTICS OF NONLINEARITY OF A SELF-OSCILLATING SYSTEM

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We suggest a method for determining the harmonic linearization coefficient of a nonlinearity in a quasi-linear self-oscillating system from the amplitude variation curve by means of a generalized interpolation. We prove a theorem which in the analytical case guarantees the uniform convergence of the interpolation process to the desired function. The determination of the dynamic characteristics of nonlinear objects was examined in [1]. Thanks to the very well developed technique for applying the harmonic linearization methd [2, 3], the determination of the harmonic linearization coefficients for nonlinear objects has a particular significance.

We consider the equation of a quasi-linear self-oscillating system

$$y^{\cdot\cdot} + \omega^2 y = \varepsilon f(y) y^{\cdot} \tag{1}$$

where $\varepsilon > 0$ is a small parameter. It is known [4] that the first approximation to the solution of (1) to within quantities of order ε^2 is $y = x \cos \psi$, where ψ is the uniformly rotating phase of the oscillations, while the amplitude of the oscillations is found from the equation

$$x^{\cdot} = \varepsilon \Phi(x) \tag{2}$$

If self-oscillations with a steady-state amplitude b are self-excited on system (1), then $x = 0$ is an unstable equilibrium position for (2), while $x = b$ is a stable one; here

$$\Phi'(0) > 0, \qquad \Phi'(b) < 0 \tag{3}$$

Let $\Phi(x)$ be a continuously differentiable function for $x \in [0, b]$ and has only simple roots, and let conditions (3)be fulfilled. Then we can write (2) as

$$x^{\cdot} = x(b - x)\varphi(x) \tag{4}$$

where the small parameter $\varepsilon > 0$ has been taken into the function φ. Moreover,

$$\varphi(x) > 0, \text{ for all } x \in [0, b] \tag{5}$$

We note that the nonlinearity in (1) is not determined statically, and $(b - x)\varphi(x)$ is the harmonic linearization coefficient for the system's nonlinearity. Having obtained experimentally a procedure for establishing self-oscillations in system (1) and assuming that the amplitude variation curve of the oscillations is the solution of (4), $x(t)$, $x(0) = x_0 \in (0, b)$ which, under condition (5), increases strictly monotonically for $t \in [0, \infty)$, we can find the right-hand side of (4)from the curve $x(t)$ by means of an approximate differentiation and a subsequent interpolation. We propose a more effective way of finding the right-hand side of (4), using the information contained in the qualitative pattern of the behavior of the solutions of (4).

Theorem . For an Eq.(4) satisfying condition (5), suppose that we know an integral curve $t(x)$, $t(x_0) = 0$, where $x_0 \in (0, b/2)$ is a small initial perturbation. Let the analytical continuation of the function $t(x)$ onto the complex plane yield a function which is regular inside an ellipse with foci at the points x_0, $b - x_0$ and with a semiaxis sum of $(b/2 - x_0)R$, where $R > 1$ is chosen in such a way that the points 0, b do not belong to this ellipse. Then, on the interval $[x_0, b - x_0]$ we can represent the function $\varphi(x)$ as

$$\varphi(x) = 1 / P_n(x) + \alpha_n(x)$$

$$P_n(x) = \sum_{i=0}^{n} d_i x^i$$

$$\alpha_n(x) \rightrightarrows 0 \quad \text{as} \quad n \to \infty, \quad x \in [x_0, b - x_0]$$

and the estimate

$$|\varphi(x) - 1 / P_n(x)| \leqslant M / \rho^{n+1} \qquad (1 < \rho < R, \quad M = \text{const})$$

is valid. Here the coefficients $P_n(x)$ are uniquely defined by the values of $t(x)$ from the interval $[x_0, b - x_0]$.

Lemma . Let $f(x)$ be an infinitely differentiable function for $x \in [0, 1]$ and let $f^{(n)}(x) \neq 0$ for all $x \in (0, 1)$, $n = 1, 2 \ldots$, while $f(0) = 0$. Then for any n the functions $f, x, x^2, \ldots, x^n$ form a Chebyshev system on the interval $(0, 1)$.

Proof. Let us prove that for any set of values $x_1, x_2, \ldots, x_{n+1}$, $x_i \in (0, 1)$, $x_i \neq x_j$ for $i \neq j$ the determinant Δ_n of order $n + 1$, whose ith row has the form

$$f(x_i)\ x_i\ x_i^2 \ldots x_i^n$$

is nonzero for any $n = 1, 2, \ldots$ We prove this by induction.

Let $\Delta_1 = 0$. Then there exist numbers λ_1, λ_2, not equal to zero, such that they serve as the coefficients of a vanishing linear combination of the columns of determinant Δ_1. We consider the function $F(x) = \lambda_1 f(x) + \lambda_2 x$. It has the three roots $0, x_1, x_2$ on the interval $[0, 1)$. By Rolle's theorem $F'(x) = \lambda_1 f(x) + \lambda_2$ has two distinct roots ξ_1, $\xi_2 \in (0, 1)$. Then, the determinant $f'(\xi_1) - f'(\xi_2) = 0$ and $f''(\xi) = 0$ for $\xi \in (\xi_1, \xi_2)$, which contradicts the hypothesis. Hence it follows that $\Delta_1 \neq 0$.

Let $\Delta_{n-1} \neq 0$. Assume, despite the lemma's assertion, that $\Delta_n = 0$. Then the relation $\lambda_1 S_1 + \ldots + \lambda_{n+1} S_{n+1} = 0$ exists between the columns S_i of the determinant Δ_n, moreover, $\lambda_1 \neq 0$, since otherwise a certain Vandermonde determinant would be zero, and $\lambda_{n+1} \neq 0$ by the inductive assumption. We consider the function $F(x) = \lambda_1 f(x) + \lambda_2 x + \ldots + \lambda_{n+1} x^n$. The points $0, x_1, \ldots, x_{n+1}$ are the roots of this function. Then $F^{(n)}(x) = \lambda_1 f^{(n)}(x) + n!\, \lambda_{n+1}$ has two roots $\xi_1, \xi_2 \in (0,1)$ and $f^{(n+1)}(\xi) = 0$ for some $\xi \in (0, 1)$. The contradiction obtained proves that $\Delta_n \neq 0$ and, together with this, the lemma.

We proceed to the proof of the theorem. We note that

$$t(x) = \int_{x_0}^{x} \frac{dz}{z(b - z)\varphi(z)}$$

and we set $1/\varphi(z) = f(z)$. It is obvious that $f(z)$, as a function of a complex variable, is regular in the same region that $t(x)$ is, since this region does not contain the points 0 and b. On the interval $[x_0, b - x_0]$ we introduce an infinite triangular matrix of Fejer interpolating nodes [5] in the following manner:

$$x_k^{(n)} = x_0 + (b/2 - x_0)[1 - \cos \pi(2k - 1)/2(n + 1)], \quad \begin{matrix} k = 1, 2, \ldots n + 1 \\ n = 1, 2, \ldots \end{matrix}$$

We construct an interpolation process with an nth-degree polynomial $P_n(x)$ by setting

$$t(x_k) = \int_{x_0}^{x_k} P_n(z) / z(b-z)\, dz, \qquad k = 1, 2, \ldots n+1$$

Here and subsequently we omit the superscripts in the node designations.

We obtain a linear system of equations in the coefficients of the polynomial $P_n(z) = d_0 + d_1 z + \ldots + d_n z^n$. The determinant of this system can be represented as the product of a constant factor by a determinant Δ_n' of order $n+1$, each ith row of which is

$$\ln x_i / x_0 \ln (b - x_0) / (b - x_i) \quad x_i - x_0 \ \ldots \ x_i^{n-1} - x_0^{n-1}$$

The determinant Δ_n' is nonzero for any n. This is verified by arguments exactly repeating those in the proof of the lemma, moreover, on the basis of the lemma, λ_1 and λ_2 are nonzero, and the determinant

$$\xi_1^{-n}(b-\xi_2)^{-n} - \xi_2^{-n}(b-\xi_1)^{-n}$$

where $\xi_1 \neq \xi_2$ and $\xi_1, \xi_2 \in (x_0, b - x_0)$, does not equal to zero. Thus, for any n the coefficients of polynomial P_n are uniquely determined by the values $t(x_k)$.

The polynomial constructed is an interpolation polynomial for $f(z)$ since

$$t(x_k) = \int_{x_0}^{x_k} f(z) / z(b-z)\, dz = \int_{x_0}^{x_k} P_n(z) / z(b-z)\, dz, \qquad k = 1, 2, \ldots n+1$$

moreover, the interpolating nodes for the function $f(z)$ lie strictly between the nodes x_k. The nodes x_k selected are Fejér nodes, therefore, the interpolating nodes for function $f(z)$ also are Fejer nodes, which follows from [5] (Lemma 1, p. 29). On the basis of the corollary to Theorem 1 (see [5], p. 36) we obtain the convergence of the interpolation polynomials P_n to the function f with the estimate $|P_n(x) - f(x)| \leqslant M_1/\rho^{n+1}$ for all $x \in [x_0, b - x_0]$, where $1 < \rho < R$, $M_1 = \text{const}$. The estimate in the theorem is obtained from the continuity and from the positiveness of $f(x)$ on the interval $[x_0, b - x_0]$.

Note. The interpolation process constructed ensures a sufficiently rapid convergence for a sufficient smoothness of the function $t(x)$ and for a special choice of the nodes. For a practical determination of the function $\varphi(x)$ it may be advisable to seek for $t(x)$ the generalized polynomial of best approximation for some fixed degree n,

$$Q_n(t; x) = \sum_{k=0}^{n} c_k \psi_k(x)$$

$$\psi_k(x) = \int_{x_0}^{x} \frac{z^k}{z(b-z)}\, dz, \qquad k = 0, 1, \ldots n$$

by minimizing some criterion for the proximity of the functions $t(x)$ and $Q_n(t; x)$ on the set of all known values of $t(x)$; here the $\psi_k(x)$ are linearly independent functions. In practice this is a series of discrete values of instants of time, and the determination of the best $Q_n(t; x)$ can be successfully effected by mathematical programing methods. The function $(b - x)(c_0 + c_1 x + \ldots + c_n x^n)^{-1}$ is an approximate representation of the harmonic linearization coefficient.

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PROGRAM AND POSITION ABSORPTION IN DIFFERENTIAL GAMES

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We investigate the relations between the position and the program absorption sets. We cite an example in which the construction of the position absorption set [1 – 3] is reduced to the determination of a finite number of program absorption sets [4, 5].

It is known that in the general case the construction of a position absorption set leads to the determination of a countable sequence of program absorption sets [1, 3, 6, 7]. Also well known are the cases when the position absorption set is determined by one program absorption operation [2, 3, 5, 8]. We consider a linear differential game of pursuit. Let the motion of a conflict-controlled system be described by the equation

$$dx / dt = A(t)x + u - v \tag{1}$$

Here x is the n-dimensional system phase vector; $A(t)$ is an $n \times n$ matrix with coefficient depending continuously on t ; u and v are the controls of the first and second players, respectively, whose realizations are constrained by $u[t] \in P_t$, $v[t] \in Q_t$, where the closed convex sets P_t and Q_t depend piecewise-continuously on t.

In the phase space R_n we are given a set M which is usually assumed closed and convex. The solution of the pursuit problem consists of having to construct the first player's strategy which guarantees that the phase point $x[t]$ is taken onto the aim set M. It is assumed that information on the game position $(t, x[t])$ realized is available to the pursuer. Thus, the pursuit strategies are certain functions $U = U(t, x)$. The classes of players' strategies, containing the solution of the position differential game, were introduced in [2, 7].

Let us briefly describe certain elements of extremal construction used in solving position